NORTH-HOLLAND

A Rough Logic Based on Incomplete Information and Its Application

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ABSTRACT

This paper discusses a new rough logic based on incomplete information. Originally rough logic was constructed upon knowledge systems of complete information. In this paper, we consider incomplete knowledge representation systems in which information functions are nondeterministic. After modifying the usual definitions of upper and lower approximation operations in rough logic, we examine some properties of these operations. Also, we consider a relationship of the proposed logic to KTB *modal logic. Then, we remark about an application of incomplete information systems to decision logic.*
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0. INTRODUCTION

The problem of knowledge representation has been one of the main topics in the study of artificial intelligence and information systems. In [7] and [8], Orlowska and Pawlak discussed theoretical foundations of knowledge representation from the point of view of logic. Pawlak [9] introduced *rough sets*, which correspond to the approximations of sets of objects by means of equivalence relations modeling indiscernibility of knowledge. Pawlak published an introductory textbook [10] on rough sets which contains basic notions of rough theory and various applications. Also, del Cerro and Orlowska [2] proposed a deductive system for indiscernibility logic. These logics are strongly related to modal logic. Starting from these

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works, in [4–6] the present author presented some logical systems called *modal fuzzy logics* to combine the concepts of "rough" and "fuzzy."

This paper discusses another approach to the logical foundations of knowledge representation problems. Our attention focuses on incomplete information such that the so-called information functions of information systems are nondeterministic. In recent years a variety of formalisms for knowledge have been developed which addresses several aspects of handling imperfect knowledge such as uncertainty, vagueness, imprecision, and fuzziness. As in my previous works, this paper is motivated by the fact that our knowledge is primitively imperfect. The concept of incomplete information in this paper came from Lipski's paper [3]. We present a new rough logic which is obtained by introducing new modal operations $\langle\ \rangle, [\ ]$ instead of the usual ones in the rough logic. After showing some properties of the operations, we consider a relationship of the logic to the Brouwersche system KTB. Finally, we remark on an application of incomplete information systems to decision logic.

1. INCOMPLETE INFORMATION SYSTEM

In this section, first we review an information system

$$K = (\mathrm{OB}, \mathrm{AT}, \{\mathrm{VAL}a\}_{a \in \mathrm{AT}}, f).$$

In this system K, OB, AT, VAL, and f are defined as follows:

1. OB is a set (not necessarily finite) of objects;
2. AT is a set (not necessarily finite) of attributes;
3. VALa is a set (not necessary finite) of values of attribute a in AT, and VAL is the union of all the sets VALa;
4. f is a mapping from OB $\times$ AT into VAL.

The function f of K just defined is called an *information function*. This system is usually called an *information system*. A simple example of an information system is given in Table 1. Here OB $= \{o_1, o_2, o_3, o_4\}$ and AT $= \{a, b, c\}$, and f is given in the table.

Table 1

f	a	b	c
o_1	1	0	1
o_2	1	1	1
o_3	2	1	2
o_4	1	0	1

We define a binary relation R on the set OB as follows:

$$(o_i, o_j) \in R \quad \text{iff} \quad f(o_i, a) = f(o_j, a) \quad \text{for each } a \in \text{AT}.$$

The relation R on the set OB is referred to as *indiscernibility* with respect to attributes from the set AT [9, 10]. It is easily shown that R is an equivalence relation on the set OB. Speaking more generally, let R be an equivalence relation defined on X. Let us denote the equivalence class of x in X in the sense of R by $[x]_R$. Given a subset S of X and an equivalence relation R, a *lower* and an *upper* approximation of S with respect to R are defined as follows:

DEFINITION 1.1 *Let S be a subset of a given set X, and R be an equivalence relation defined on X. Then $R_*(S)$ and $R^*(S)$ are defined as follows*:

$$R_*(S) = \{x \in X | [x]_R \subseteq S\}, \qquad R^*(S) = \{x \in X | [x]_R \cap S \neq \emptyset\}.$$

In the definition above of an information system, f was a mapping from OB $\times$ AT into VAL. Here, we modify this information function as follows:

4. f is a mapping from OB $\times$ AT into 2^{VAL}.

The resulting information system is called an *incomplete information system*. A simple example of an incomplete information system is given in Table 2, where $f(o_4, a) = 1$ or 2, and $f(o_2, b) = \emptyset$ (means "undefined").

2. NEW ROUGH LOGIC

We propose a new rough logic with incomplete information (denoted by INCRL) whose syntax and semantics are described as follows: Consider four disjoint classes of symbols:

1. Propositional symbols: $p, q, \ldots, p_1, p_2, \ldots,$
2. Propositional operations: $\neg$ (negation), $\wedge$ (conjunction), (disjunction), $\supset$ (implication), $\equiv$ (equivalence).
3. Modal operations: [ ] (surely), and $\langle\ \rangle$ (possibly).

Table 2

f	a	b	c
o_1	1	0	1
o_2	1	—	1
o_3	2	1	2
o_4	1, 2	0	1

4. Brackets: (,).

The set FOR of all well-formed formulas (for short, wff's) of INCRL is the least set satisfying the following conditions:

1. The set of propositional symbols is a subset of FOR.

2. $A, B \in$ FOR implies $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B) \in$ FOR.

3. $A \in$ FOR implies $([\,]A), (\langle\,\rangle A) \in$ FOR.

In the standard rules, brackets in wff's may be omitted. Wff's are usually denoted by $A, B, F, G, \ldots$. As seen later, the modal operations $[\,]$ and $\langle\,\rangle$ are considered as the lower approximation operation and the upper approximation operation, respectively.

To give the semantics of INCRL, we introduce the concept of completion of an incomplete information system. Let K be an incomplete information system; then we construct a complete information system K' such that only one value is selected from the set of values of f. In this case, we assume that an extraordinary symbol ϵ is selected from $\varnothing$. Of course, there are many possibilities of selection. For example, a completion of the incomplete information system of Table 2 is given in Table 3.

When K' is a completion of K, we write $K' \geq K$. The completion means that our knowledge becomes exact as we get new information.

By a *model*, we mean an incomplete information system

$$K = (\mathrm{OB}, \mathrm{AT}, \{\mathrm{VAL}a\}_{a \in \mathrm{AT}}, f)$$

and a valuation function v_K, where

OB is a nonempty set of objects,

AT is a nonempty set of attributes,

f is a mapping from OB $\times$ AT into 2^{VAL},

Table 3

f	a	b	c
o_1	1	0	1
o_2	1	ϵ	1
o_3	2	1	2
o_4	1	3	2

v_K is a valuation function that is recursively defined as follows: For a propositional symbol p, $v_K(p)$ is a subset of OB. For wff's,

$$
\begin{aligned}
v_K(\neg F) &= -v_K(F),\\
v_K(F \wedge G) &= v_K(F) \cap v_K(G),\\
v_K(F \vee G) &= v_K(F) \cup v_K(G),\\
v_K(F \supset G) &= v_K(\neg F \vee K),\\
v_K(F \equiv G) &= v_K((F \supset G) \wedge (G \supset F)),\\
v_K([\,]F) &= \bigcap_{K' \geq K} R(K')_* v_K(F),\\
v_K(\langle\,\rangle F) &= \bigcup_{K' \geq K} R(K')^* v_K(F).
\end{aligned}
$$

In the above definition, K' is a completion of K, and then $R(K')$ means an equivalence relation determined by K'. Further, $R(K)_*$ and $R(K)^*$ correspond to R_* and R^* of a complete information system mentioned in Section 2, respectively. The definitions of $v_K([\,]F)$ and $v_K(\langle\,\rangle F)$ mean that the present value of $f(o_i, a)$ is not uniquely determined but is eventually determined by increasing of our knowledge.

Here, we note that Yao et al. [11] define lower and upper approximations for incomplete information systems. Their definition is interesting, and it is equivalent to the above one except in the treatment of an object whose attribute value is null.

Given a model K, we say that a formula F is *satisfiable* by an object o in the model K (denoted by $K, o \models F$) iff $o \in v_K(F)$ holds. Further, $\text{ext}_K F$ is defined as follows:

$$\text{ext}_K F = \{o \in \text{OB} \mid K, o \models F\}.$$

Let Γ be a set of wff's, and K, o be a model and an object, respectively. Then Γ is said to be *satisfiable* iff $K, o \models F$ for all $F \in \Gamma$. When there is no confusion, $x \in \text{ext}_K F$ is abbreviated to $x \in F$. Then a wff F is called *true in a model* K (denoted by $\models KF$) iff $\text{ext}_K F = \text{OB}$. Also, a wff F is called *valid* (denoted by $\models F$) iff F is true in every model.

3. PROPERTIES OF INCRL

In this section, we will give some properties of INCRL.

FACT 3.1

(1) $\models [\,]F \supset F$, $\models F \supset \langle\,\rangle F$.

(2) If $\models F \supset G$ then $\models [\,]F \supset [\,]G$ and $\models \langle\,\rangle F \supset \langle\,\rangle G$.

(3) $\models [\,]F \equiv \neg\langle\,\rangle\neg F$, $\models \langle\,\rangle F \equiv \neg[\,]\neg F$.

(4) $\models \langle\,\rangle(F \wedge G) \supset \langle\,\rangle F \wedge \langle\,\rangle G.$
(5) $\models \langle\,\rangle(F \vee G) \equiv \langle\,\rangle F \vee \langle\,\rangle G.$
(6) $\models [\,](F \wedge G) \equiv [\,]F \wedge [\,]G.$
(7) $\models [\,]F \vee [\,]G \supset [\,](F \vee G).$
(8) $\models F \supset [\,]\langle\,\rangle F.$

Proof Let K be an arbitrary information system.

(1): $v_K([\,]F)$ was defined as follows:

$$\bigcap_{K' \geq K} R(K')_* v_K(F).$$

But, for every complete information system K' of K, $R(K')_* v_K(F) \subseteq v_K(F)$. Therefore, we have

$$\bigcap_{K' \geq K} R(K')_* v_K(F) \subseteq v_K(F).$$

Thus, we have $\models [\,]F \supset F$. The proof of $\models F \supset \langle\,\rangle F$ is similar.

(2): Let K_i be an arbitrary completion information system of K. Then it is well known that from $\models F \supset G$ we get $R(K_i)_* v_K(F) \subseteq R(K_i)_* v_K(G)$. Therefore, we have

$$\bigcap_{K' \geq K} R(K_i)_* v_K(F) \subseteq \bigcap_{K' \geq K} R(K_i)_* v_K(G).$$

Thus, we have $\models [\,]F \supset [\,]G$. The proof of $\models \langle\,\rangle F \supset \langle\,\rangle G$ is similar.

(3):

$$\begin{aligned} v_K(\neg\langle\,\rangle\neg F) &= -v_K(\langle\,\rangle\neg F) \\ &= -\Big(\bigcup_{K' \geq K} R(K')^* v_K(\neg F)\Big) \\ &= \bigcap_{K' \geq K} R(K')_* - v_K(\neg F) \\ &= \bigcap_{K' \geq K} R(K')_* v_K(F). \end{aligned}$$

Therefore, we get $\models [\,]F \equiv \neg\langle\,\rangle\neg F$.

The proof of $\models \langle\,\rangle F \equiv \neg[\,]\neg F$ is similar.

(4): We have the following:

$$\begin{aligned} v_K(\langle\,\rangle(F \wedge G)) &= \bigcup_{K' \geq K} R(K')^* v_K(F \wedge G) \\ &= \bigcup_{K' \geq K} R(K')^*(v_K(F) \cap v_K(G)) \\ &\subseteq \bigcup_{K' \geq K} R(K')^* v_K(F) \cap \bigcup_{K' \geq K} (R(K')^* v_K(G)). \end{aligned}$$

Therefore, we get $\models \langle\,\rangle(F \wedge G) \supset \langle\,\rangle F \wedge \langle\,\rangle G.$

(5): We have the following:

$$\begin{aligned} v_K(\langle\ \rangle(F \vee G)) &= \bigcup_{K' \geq K} R(K')^* v_K(F \wedge G) \\ &= \bigcup_{K' \geq K} R(K')^*(v_K(F) \cup v_K(G)) \\ &= \bigcup_{K' \geq K} R(K')^* v_K(F) \cup \bigcup_{K' \geq K} (R(K')^* v_K(G)). \end{aligned}$$

Therefore, we get $\models \langle\ \rangle(F \vee G) \equiv \langle\ \rangle F \vee \langle\ \rangle G$.

(6): By making use of (3), (5), we easily get (6).

(7): By making use of (3), (4), we easily get (7).

(8): Let us assume that $x \in v_K(F)$ and $(x, y) \in R(K')$ for some completion K' of K. In this case, we have $(y, x) \in R(K')$. Therefore, we have

$$x \in \bigcap_{K' \geq K} R(K')_* \bigcup_{K' \geq K} R(K')^* v_K(F).$$

Thus, we have (8). ■

It is well-known that modal logics are characterized in terms of Kripke model (W, R, V). They are characterized by the properties of the relation R corresponding to the modal operation of the system. From this point of view, it is interesting that we have (1) of Fact 3.1. characterizing reflexivity as well as (8) representing symmetry. But, $\langle\ \rangle\langle\ \rangle F \supset \langle\ \rangle F$, which characterizes transitivity, is not valid in our logic. This is shown in the following incomplete information system:

	a	b
o_1	1	0
o_2	1, 2	0
o_3	2, 3	0
o_4	1, 2	0

We consider o_3 as $v_K(p)$. Then

$$o_1 \in \bigcup_{K' \geq K} R(K')^* \bigcup_{K' \geq K} R(K')^* v_K(p),$$

but

$$o_1 \notin \bigcup_{K' \geq K} R(K')^* v_K(p).$$

Hence we know that the proposed rough logic can be considered as a kind of KTB modal logic.

4. RELATIONSHIPS OF INCRL TO THE BROUWERSCHE SYSTEM

In this section, we consider the relationships of INCRL to the Brouwersche system KTB. It is shown that all provable wff 's KTB are valid in INCRL. This is shown in the following Fact 4.1. The modal system KTB consists of the following axioms and rules:

Axioms.

(Ax1) The axioms of the classical propositional logic.

(Ax2) (i) $[\,](A \supset B) \supset ([\,]A \supset [\,]B)$,
(ii) $[\,]A \supset A$,
(iii) $A \supset [\,]\langle\,\rangle A$.

Rules of inference.

(*Modus ponens*): From A and $A \supset B$, we get B.
Generalization: From A, we get $[\,]A$.

FACT 4.1 *All provable wff's of* KTB *are valid in* INCRL.

Proof Obviously, every wff of (Ax1) is valid in INCRL. Further, it has been already shown in Fact 3.1 that (i), (ii), (iii) of (Ax2) are valid in INCRL. Also, it is easily shown that the validity in INCRL is preserved in the rules of inference.

Then, we have the result. ∎

We shall now define a Kripke model $\langle W, R, V\rangle$ from the model of INCRL. As the set W of worlds, we consider the set OB of objects. Then we define the relation R over W as follows:

$$(o_i, o_j) \in R \quad \Leftrightarrow \quad o_i \text{ and } o_j \text{ are indiscernible in a completion } K^\lambda \text{ of } K.$$

Also, V is defined in the usual way, i.e,. $V(p, o_i)$ is 1 or 0, according as $o_i \in v(p)$ or not.

FACT 4.2 *The R defined above is reflexive and symmetric.*

Proof From the definitions of indiscernibility, this fact is obvious. ∎

But we have the following fact:

FACT 4.3 *The R defined above is not transitive.*

Proof This is shown by using again the example in preceding section:

K	a	b
o_1	1	0
o_2	1, 2	0
o_3	2, 3	0
o_4	1, 2	0

In this system K, $(o_1, o_2) \in R$ and $(o_2, o_3) \in R$, where K^1 and K^2 are completions of K as follows:

K^1	a	b
o_1	1	0
o_2	1	0
o_3	3	0
o_4	2	0

K^2	a	b
o_1	1	0
o_2	2	0
o_3	2	0
o_4	1	0

But $(o_1, o_3) \notin R$, since o_1 and o_3 are discernible in any completion of K. ■

By making use of this relation R, $V(\langle\,\rangle F, o)$ and $V([\,]F, o)$ are defined in the standard way, i.e.,

$$V(\langle\,\rangle F, o) = \max_{(o, o_i) \in R} (V(F, o_i)),$$

and $V([\,]F, o) = V(\neg \langle\,\rangle \neg F, o)$. Then we have the following fact:

FACT 4.4 *A wff F is valid in* INCRL *iff F is so in the proposed Kripke model.*

Proof For the proof, it is sufficient to show

$$o \in \bigcup_{K' \geq K} R(K')^* v_K(F) \quad \Leftrightarrow \quad V(\langle\,\rangle F, o) = 1.$$

But this is shown without difficulty from the definition $o \in \bigcup_{K' \geq K} R(K')^* v_K(F)$ and $V(\langle\,\rangle F, o) = 1$. ■

Here, we notice the following fact: KTB is complete with respect to the class of finite Kripke models that are reflexive and symmetric (for the proof, see [1]). By making use of this fact, it is seen that INCRL is axiomatizable. We leave the detailed mathematical proof to another paper.

5. APPLICATION TO DECISION LOGIC

The decision logic based on complete information has been well discussed (for example, see [10]). But the same has been not yet done for the decision logic of incomplete information systems. In the final section of this paper, we shall consider an application of INCRL to decision logic.

Let $K = (\text{OB}, \text{AT}, \{\text{VAL}\, a\}_{a \in \text{AT}}, f)$ be a complete information system, and $C, D \subseteq \text{AT}$ be two subsets of attributes, called *condition* and *decision attributes*, respectively. In the same way as for the usual decision logic, let us assume here that OB, AT, and VAL are finite sets. For example, in Table 1, let C be $\{a, b\}$ and D be $\{c\}$. Here, we use the notation $v(a) = 1$ to represent that the value of attribute a is 1. Then from Table 1 we get $v(c) = 1$ from $v(a) = 1$ and $v(b) = 0$. Similarly, from $v(a) = 2$ and $v(b) = 1$ we get $v(c) = 2$. In Table 1, the values of the condition attributes determine uniquely the value of the decision attribute. But, an information system can be inconsistent even if it is complete, in that different values of decision attributes come from the same values of condition atributes. The exact definition of decision logic has been given in [10]; it is a kind of prescription, which specifies what decisions (actions) should be undertaken when some conditions are satisfied. In the incomplete case, this concept is extended to possible (or absolute) prescriptions.

Let A and B be wff's to represent the condition and the decision of a row in K, respectively. Then $A \supset B$ is called a *decision rule*. For example, in Table 1, $(a, 1) \wedge (b, 0) \supset (c, 1)$ is a decision rule, where $(a, 1)$ means $v(a) = 1$, and so on. In an information system K, if a decision rule $A \supset B$ is true in K, then the decision rule is said to be *consistent* in K. Further, if a conjunction of all decision rules obtained from K is true in K, then K itself is said to be consistent. The consistency of K means that we can consistently decide one action depending the condition shown in K.

We will now extend the above-mentioned decision logic to the rough logic connected with incomplete information.

Modifying the definitions in decision logic [10], we consider the *characteristic formula* of an incomplete information system. This is built from propositional symbols to represent conditions and decision. For example, for the incomplete information system shown in Table 4, the characteristic formula is as follows:

$$((a,\{1\}) \wedge (b,\{0,1\}) \supset (c,\{1\})) \wedge ((a,\{1\}) \wedge (b,\varnothing) \supset (c,1))$$

$$\wedge((a,\{2\}) \wedge (b,\{1\}) \supset (c,\{2\})) \wedge ((a,\{1,2\}) \wedge (b,\{0\}) \supset (c,1)). \quad \text{(I)}$$

Table 4

	a	b	c
o_1	1	0, 1	1
o_2	1	—	1
o_3	2	1	2
o_4	1, 2	0	1

For the characteristic formula (I), we consider the following expression:

$$\mathscr{P}[((a,\{1\}) \wedge (b,\{0,1\}) \supset (c,\{1\})) \wedge ((a,\{1\}) \wedge (b,\varnothing) \supset (c,1)) \wedge ((a,\{2\}) \wedge (b,\{1\}) \supset (c,\{2\})) \wedge ((a,\{1,2\})(b,\{0\}) \supset (c,1))]. \quad \text{(II)}$$

(II) is generally called the *characteristic formula of possibility* (abbreviated PCF) of the incomplete information system. Here, the symbol $\mathscr{P}$ means that the possibility of the characteristic formula should be considered in various completions of the corresponding incomplete information system. That is, (II) is transformed to the following formula:

$$\begin{aligned}
&[((a,\{1\}) \wedge (b,\{0\}) \supset (c,\{1\})) \wedge ((a,\{1\})(b,\varnothing) \supset (c,1))\\
&\quad \wedge ((a,\{2\}) \wedge (b,\{1\}) \supset (c,\{2\})) \wedge ((a,\{1\}) \wedge (b,\{0\}) \supset (c,1))]\\
&\vee [((a,\{1\}) \wedge (b,\{1\}) \supset (c\{1\})) \wedge ((a,\{1\})(b,\varnothing) \supset (c,1))\\
&\quad \wedge ((a,\{2\}) \wedge (b,\{1\}) \supset (c,\{2\})) \wedge ((a,\{1\}) \wedge (b,\{0\}) \supset (c,1))]\\
&\vee [((a,\{1\}) \wedge (b,\{0\}) \supset (c,\{1\})) \wedge ((a,\{1\})(b,\varnothing) \supset (c,1))\\
&\quad \wedge ((a,\{2\}) \wedge (b,\{1\}) \supset (c,\{2\})) \wedge ((a,\{2\}) \wedge (b,\{0\}) \supset (c,1))]\\
&\vee [((a,\{1\}) \wedge (b,\{1\}) \supset (c,\{1\})) \wedge ((a,\{1\})(b,\varnothing)(c,1))\\
&\quad \wedge ((a,\{2\}) \wedge (b,\{1\}) \supset (c,\{2\}))\\
&\quad \wedge ((a,\{2\}) \wedge (b,\{0\}) \supset (c,1))]. \quad \text{(III)}
\end{aligned}$$

Then we have the following fact:

FACT 5.1 *If the PCF for an incomplete information system K is true in K, then K is possibly consistent.*

Fact 5.1 has the following meaning: The possible consistency of the incomplete information system K means that we can consistently select our action depending the condition shown in K. Of course, if the PCF for an incomplete information system H is not true in H, then we cannot consistently select one action from the condition satisfying H.

In the above discussion, we considered $\mathscr{P}$ as the possibility. In the dual way, we can propose an operation $\mathscr{A}$ which is defined as $\neg \mathscr{P} \neg$. It means that the corresponding formula expresses consistency for *every* completion of the incomplete information system K.

An element of 2^{VAL} is called a *generalized value* (see [7]), and generalized values are denoted by $s_1, s_2, \ldots$. Then the following Fact 5.2 and Fact 5.3 are easily obtained. For simplicity, we assume here the decision attribute is one. The general case can be discussed in the same way.

FACT 5.2 *Let us consider two decision rules i, j. Assume that the intersection of the generalized values s_i and s_j for the decision attribute of the two rules i, j is not empty. Then the two decision rules are possibly consistent.*

FACT 5.3 *Let s_i and s_j be the same as in Fact 5.2. Also assume that the all intersections of generalized values of every condition attribute of decision rules i and j are nonempty. Then, if the symmetric difference of s_i and s_j is not empty, the two rules i and j are possibly inconsistent.*

Fact 5.3 means that if a decision table H contains two possibly inconsistent rules, then $\mathscr{A}$[characteristic formula of H] is not true in H.

It seems that there are many interesting problems concerning decision logic based on incomplete information system. Among them are the reduction of decision rules and the simplification of decision tables. We will discuss these problems in detail in further papers.

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